Multiscale Analysis and Computation for Incompressible Flow

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- The understanding of scale interactions for 3-D incompressible Euler and Navier-Stokes eqns have been a major challenge.

- For high Reynolds number flows, the degrees of freedom are so high that it is almost impossible to resolve all small scales by DNS.

- Deriving an effective equation for the large scale solution is very useful in engineering applications.

- The main difficulty in deriving effective equations is the lack of scale separation. There is a continuous spectrum of scales.

- Can homogenization theory based on scale separation produce any useful result?

- **The purpose** of this study is to derive a nonlinear homogenization result for the incompressible Euler equations.

- The nonlinear and nonlocal nature of Euler equations makes it difficult to construct a properly-posed multiscale solution.

- The key idea in deriving a multiscale solution is to use the Lagrangian description.

- The multiscale structure of the solution provides a critical guideline in constructing a general multiscale numerical method.

Formulation

We consider the 3-D incompressible Euler eqns

$$\mathbf{u}_t^\epsilon + (\mathbf{u}^\epsilon \cdot \nabla)\mathbf{u}^\epsilon + \nabla p^\epsilon = 0, \quad (1)$$

$$\nabla \cdot \mathbf{u}^\epsilon = 0, \quad (2)$$

with multiscale initial data
$\mathbf{u}^\epsilon(\mathbf{x}, 0) = \mathbf{U}(\mathbf{x}) + \mathbf{W}(\mathbf{x}, \frac{\mathbf{x}}{\epsilon})$.

$\mathbf{u}^\epsilon(t, \mathbf{x})$ and $p^\epsilon(t, \mathbf{x})$ are velocity and pressure. We assume that $\mathbf{U}$ and $\mathbf{W}$ are smooth, $\mathbf{W}(\mathbf{x}, \mathbf{y})$ is periodic in $\mathbf{y}$ and has mean zero.

- The question of interest is how to derive an effective (or homogenized) equations for the averaged velocity field as $\epsilon \to 0$.

Previous work by MPP

- The homogenization of the Euler equations with oscillating data was first studied by McLaughlin-Papanicolaou-Pironneau in 1985.

- To construct a multiscale expansion for the Euler equations, they made the assumption that the oscillation is convected by the mean flow.

$$\mathbf{u}^\epsilon(\mathbf{x},t) = \bar{\mathbf{u}}(\mathbf{x},t) + \mathbf{w}(\mathbf{x},t,\frac{\theta(\mathbf{x},t)}{\epsilon},\frac{t}{\epsilon}) + \epsilon\mathbf{u}_1(\cdot\cdot,\frac{\theta(\mathbf{x},t)}{\epsilon},\frac{t}{\epsilon}) +$$

$$p^\epsilon(\mathbf{x},t) = \bar{p}(\mathbf{x},t) + \pi(\mathbf{x},t,\frac{\theta(\mathbf{x},t)}{\epsilon},\frac{t}{\epsilon}) + \epsilon p_1(\cdot\cdot,\frac{\theta(\mathbf{x},t)}{\epsilon},\frac{t}{\epsilon}) +$$

where $\mathbf{w}(\mathbf{x},t,\mathbf{y},\tau)$, $\mathbf{u}_1(\mathbf{x},t,\mathbf{y},\tau)$, π, and p_1 are assumed to be periodic in $\mathbf{y}$ and τ, and θ is convected by the mean velocity field $\bar{\mathbf{u}}$

$$\frac{\partial\theta}{\partial t} + \bar{\mathbf{u}}\cdot\nabla_{\mathbf{x}}\theta = 0, \quad \theta(\mathbf{x},0) = \mathbf{x}\ . \tag{3}$$

Previous work – continued

• Using multiscale expansion techniques, MPP obtained periodic cell problems for $\mathbf{w}(\mathbf{x}, t, \mathbf{y}, \tau)$, $\mathbf{u}_1(\mathbf{x}, t, \mathbf{y}, \tau)$, π, and p_1.

• However, it is **not** clear whether the resulting cell problems for $\mathbf{w}(\mathbf{x}, t, \mathbf{y}, \tau)$, etc have solution that is **periodic** in both $\mathbf{y}$ and τ.

• Additional assumptions were imposed on the solution of the cell problems in order to derive a variant of the $k - \epsilon$ model.

• A more accurate ansatz which accounts for the oscillation in the characteristics may be needed.

• Our study shows that the oscillation is actually convected by the oscillatory velocity field:

$$\frac{\partial \theta^\epsilon}{\partial t} + \mathbf{u}^\epsilon \cdot \nabla_{\mathbf{x}} \theta^\epsilon = 0, \quad \theta^\epsilon(\mathbf{x}, 0) = \mathbf{x} \, . \qquad (4)$$

• This becomes obvious when we formulate the 2-D Euler equations in vorticity form

$$\omega(\mathbf{x}, t) = \omega_0(\theta^\epsilon(\mathbf{x}, t), \frac{\theta^\epsilon(\mathbf{x}, t)}{\epsilon}).$$

• It is not clear what is the multiscale structure of $\theta^\epsilon(\mathbf{x}, t)$ since its structure is coupled to the multiscale structure of $\mathbf{u}^\epsilon$.

• Further, if we formally expand $\theta^\epsilon(\mathbf{x}, t)$ into

$$\theta^\epsilon(\mathbf{x}, t) = \theta(\mathbf{x}, t) + \epsilon \theta_1(\mathbf{x}, t, \frac{\mathbf{x}}{\epsilon}, \frac{t}{\epsilon}) + \cdots .$$

then θ_1 would have $O(1)$ contribution to $\bar{\mathbf{u}}$.

Lagrangian description of the Euler equations

- The key idea in constructing multiscale solutions for the Euler equation is to reformulate the problem using θ^ϵ as a new variable.

- The multiscale structure of the solution becomes very apparent in terms of θ^ϵ variable.

- This amounts to using a Lagrangian description of the Euler equations.

- Specifically, we introduce a change of variable from $\mathbf{x}$ to θ: $\quad \theta = \theta^\epsilon(\mathbf{x}, t)$.

- It is easy to see that the inverse of this map, denoted as $\mathbf{x} = \mathbf{X}(\theta, t)$, is the flow map:

$$\frac{\partial \mathbf{X}(\theta, t)}{\partial t} = \mathbf{u}^\epsilon(\mathbf{X}(\theta, t), t), \quad \mathbf{X}(\theta, 0) = \theta.$$

• In terms of θ variable, the vorticity has a simple expression:

$$\omega^\epsilon(\mathbf{X}^\epsilon(\theta,t),t) = \frac{\partial \mathbf{X}^\epsilon}{\partial \theta}(\theta,t)\omega_0(\theta,\frac{\theta}{\epsilon}) \ ,$$

where $\omega^\epsilon = \nabla \times \mathbf{u}^\epsilon$, and ω_0 is the initial vorticity.

• Velocity can be computed via the stream function, ψ, $\mathbf{u}^\epsilon = \nabla \times \psi$, and ψ satisfies

$$-\triangle_{\mathbf{x}}\, \psi^\epsilon = \omega^\epsilon \ .$$

• In terms of the θ variable, we have

$$-\nabla_\theta \cdot \mathcal{A}\mathcal{A}^T \nabla_\theta \psi^\epsilon = \frac{\partial \mathbf{X}^\epsilon}{\partial \theta}(\theta,t)\omega_0(\theta,\frac{\theta}{\epsilon}) \ ,$$

where $\mathcal{A} = \frac{\partial \theta}{\partial \mathbf{x}} = \left(\frac{\partial \mathbf{x}}{\partial \theta}\right)^{-1}$.

• Using $|\frac{\partial \mathbf{x}}{\partial \theta}| = 1$, we can express $\mathcal{A}$ in terms of $\frac{\partial \mathbf{x}}{\partial \theta}$.

Multiscale expansion of $\mathbf{X}$ and ψ

• We are mainly interested in obtaining an averaged equation for the well-mixing long time solution of the Euler equation.

• For this reason, we look for multiscale solutions of the following form:

$$\psi^{\epsilon} = \psi^{(0)}(t, \theta) + \epsilon \psi^{(1)}(t, \theta, \tau, \mathbf{y}) + O(\epsilon^2) \ ,$$

$$\mathbf{X}^{\epsilon} = \mathbf{X}^{(0)}(t, \theta) + \epsilon \mathbf{X}^{(1)}(t, \theta, \tau, \mathbf{y}) + O(\epsilon^2),$$

where $\psi^{(1)}$ and $\mathbf{X}^{(1)}$ are periodic functions with respect to $\mathbf{y}$. Here $\mathbf{y} = \theta/\epsilon$ and $\tau = t/\epsilon$.

Homogenized Equations

It can be shown that $\mathbf{X}^{(0)}$, $\mathbf{X}^{(1)}$, and $\psi^{(0)}$, $\psi^{(1)}$ satisfy the following homogenized equations:

$$\partial_t \mathbf{X}^{(0)} - \left(\nabla_\alpha^\perp \psi^{(0)} \cdot \nabla_\alpha\right) \mathbf{X}^{(0)} = 0, \quad \mathbf{X}^{(0)}|_{t=0} = \alpha,$$

$$\partial_\tau \mathbf{X}^{(1)} - \left(\nabla_\alpha \mathbf{X}^{(0)} + \nabla_\mathbf{y} \mathbf{X}^{(1)}\right) \nabla_\mathbf{y}^\perp \psi^{(1)} = 0, \quad \mathbf{X}^{(1)}|_{\tau=0} = \mathbf{0},$$

and

$$\nabla_\alpha^\perp \cdot \left(\nabla_\alpha X^{(0)^\top} \nabla_\alpha X^{(0)} \nabla_\alpha^\perp \psi^{(0)}\right) + \nabla_\alpha^\perp \cdot \langle \mathcal{A}_0^\top \mathcal{A}_0 \nabla_\mathbf{y}^\perp \psi^{(1)} \rangle = \nabla_\alpha^\perp$$

$$\nabla_\mathbf{y}^\perp \cdot \left(\mathcal{A}_0^\top \mathcal{A}_0 \nabla_\mathbf{y}^\perp \psi^{(1)}\right) = \nabla_\mathbf{y}^\perp \cdot \mathbf{W},$$

- Here $\mathcal{A}_0 = \nabla_\theta \mathbf{X}^{(0)} + \nabla_\mathbf{y} \mathbf{X}^{(1)}$. It can be shown that $|\mathcal{A}_0| \equiv 1$, implying well-posedness of ψ^1 eqn.

- Equation for $\mathbf{X}^1$ requires a solvability condition.

Solvability Condition and Projection

- Solvability condition for $\mathbf{X}^{(1)}$ requires that $\epsilon\mathbf{X}^{(1)}(\cdot,\cdot,\frac{\theta}{\epsilon},\frac{t}{\epsilon}) \to 0$ as $\epsilon \to 0$.

- Let $\mathbf{u}_1 = \nabla_{\mathbf{y}}\psi^{(1)}$ be the cell velocity, and $\mathbf{Y}(\mathbf{y},\tau)$ be the cell characteristic.

- We decompose the cell velocity field into two parts:

$$\mathbf{u}_1(\theta,t,\mathbf{y},\tau) = \mathbf{u}_{11}(\theta,t,\mathbf{y},\tau) + \mathbf{u}_{12}(\theta,t,\mathbf{y},\tau).$$

- We need to remove the non-mixable part $\mathbf{u}_{12}$

$$\frac{1}{T}\int_0^T \mathbf{u}_{12}(\theta,t,\mathbf{Y}(\mathbf{y},\tau),\tau)d\tau \to \alpha \neq 0$$

as $T \to \infty$.

Solvability – continued

- For Navier-Stokes equations, the viscosity and random forcing play the role to eliminate the non-mixable component of the flow velocity.

- Eliminating this non-mixable component is essential for the flow to be fully mixed, and to reveal certain universality and scale similarity.

- For inviscid Euler equations without external forcing, the projection step provides a systematic way to eliminate the non-mixable component.

- It can be also viewed as an acceleration method for the flow to be fully mixed.

Multiscale analysis in Eulerian frame

• Since small scale is propagated along the flow map, it is natural to look for multiscale velocity and pressure in the following form:

$$\mathbf{u}^\epsilon(t,\mathbf{x}) = \mathbf{u}(t,\mathbf{x},\tau) + \mathbf{w}(t,\mathbf{x},\tau,\mathbf{y}) + O(\epsilon),$$
$$p^\epsilon(t,\mathbf{x}) = p(t,\mathbf{x},\tau) + q(t,\mathbf{x},\tau,\mathbf{y}) + O(\epsilon),$$

where $\tau = t/\epsilon$ and $\mathbf{y} = \theta^\epsilon(t,\mathbf{x})/\epsilon$, and θ^ϵ is defined implicitly by

$$\theta^\epsilon = \theta(t,\mathbf{x}) + \epsilon\theta^{(1)}(t,\mathbf{x},\frac{t}{\epsilon},\frac{\theta^\epsilon}{\epsilon}) + O(\epsilon^2),$$

and satisfies the following evolution equation:

$$\theta^\epsilon_t + \mathbf{u}^\epsilon \cdot \nabla\theta^\epsilon = \mathbf{0}, \quad \theta^\epsilon(0,\mathbf{x}) = \mathbf{x}.$$

We assume that $\mathbf{w}$, and q have zero mean with respect to $\mathbf{y}$.

• The multiscale structure of the velocity field $\mathbf{u}^\epsilon(t, \mathbf{x})$ is now coupled to the small scale structure of $\theta^\epsilon(t, \mathbf{x})$.

• Naively, one may be tempted to assume that θ^ϵ has the following multiscale scale expansion:

$$\theta^\epsilon(t, \mathbf{x}) = \theta(t, \mathbf{x}) + \epsilon\theta^{(1)}(t, \mathbf{x}, \frac{t}{\epsilon}, \frac{\mathbf{x}}{\epsilon}) + O(\epsilon^2).$$

• This is wrong. Under this assumption, θ^ϵ could develop infinitely many scales in powers of ϵ for $t > 0$.

• It is important to define θ implicitly through

$$\theta^\epsilon = \theta(t, \mathbf{x}) + \epsilon\theta^{(1)}(t, \mathbf{x}, \frac{t}{\epsilon}, \frac{\theta^\epsilon}{\epsilon}) + O(\epsilon^2),$$

which has the same small scale structure as $\mathbf{u}^\epsilon$.

Define $\nabla_x \theta^\epsilon = \mathcal{B}^{(0)} + \epsilon \mathcal{B}^{(1)}$. It can be shown that

$$\mathcal{B}^{(0)} = (\mathcal{I} - D_y \theta^{(1)})^{-1} D_x \theta;$$

$$\mathcal{B}^{(1)} = (\mathcal{I} - D_y \theta^{(1)})^{-1} D_x \theta^{(1)}.$$

Substituting the expansion into the Euler equations, we get

$$\frac{1}{\epsilon}[\ \partial_\tau \mathbf{w} + \partial_\tau \mathbf{u} + {\mathcal{B}^{(0)}}^\top \nabla_y q\ ] + \partial_t \mathbf{u} + \partial_t \mathbf{w} +$$
$$((\mathbf{u} + \mathbf{w}) \cdot \nabla_x)(\mathbf{u} + \mathbf{w}) + {\mathcal{B}^{(1)}}^\top \nabla_y q + \nabla_x (p + q) = 0.$$

This leads to the following result:

$$\begin{aligned}
\epsilon^{-1} &: \quad \partial_\tau \mathbf{w} + \partial_\tau \mathbf{u} + {\mathcal{B}^{(0)}}^\top \nabla_y q = \mathbf{0}, \\
\epsilon^{0} &: \quad \partial_t (\mathbf{u} + \mathbf{w}) + ((\mathbf{u} + \mathbf{w}) \cdot \nabla_x (\mathbf{u} + \mathbf{w}) \\
&\qquad\qquad + \nabla_x (p + q) + {\mathcal{B}^{(1)}}^\top \nabla_y q = 0.
\end{aligned}$$

Further, since $\mathbf{w}$ and q have zero mean, we obtain by averaging the $\mathbf{u}$-equation w.r.t. $\mathbf{y}$

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla_x \mathbf{u} + \langle (\mathbf{w} \cdot \nabla_x) \mathbf{w} \rangle + \langle {\mathcal{B}^{(1)}}^\top \nabla_y q \rangle = -\nabla_x p,$$

Using the weak formulation, we can show that

$$\langle {\mathcal{B}^{(1)}}^\top \nabla_y q \rangle = \langle \mathbf{w} \nabla_x \cdot \mathbf{w} \rangle \ .$$

Thus the homogenized equation for $\mathbf{u}$ becomes

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla_x \mathbf{u} + \nabla_x \cdot \langle \mathbf{w}\mathbf{w} \rangle = -\nabla_x p,$$

$$\nabla \cdot \mathbf{u} = 0.$$

$\mathcal{B}^{(0)}$ is determined from

$$\mathcal{B}^{(0)} = (\mathcal{I} - D_y\theta^{(1)})^{-1}D_x\theta;$$

$\theta(t, \mathbf{x})$ is convected by the mean velocity.

$$\partial_t\theta + (\mathbf{u}\cdot\nabla_x)\theta = 0, \quad \theta|_{t=0} = \mathbf{x};$$

$\theta^{(1)}(t, \mathbf{x}, \tau, \mathbf{y})$ as function of $(\tau, \mathbf{y})$ is evolved by

$$\partial_\tau\theta^{(1)} + (\mathbf{w}\cdot\nabla_x)\theta = \mathbf{0}, \quad \theta^{(1)}|_{\tau=0} = \mathbf{0};$$

Again, we use a projection method to ensure no secular growth in $\epsilon\theta^{(1)}$

$$\mathbf{w} \;\leftarrow\; \mathbf{w} - \lim_{T\to\infty}\frac{1}{T}\int_0^T \mathbf{w}\, d\tau.$$

Extension to non-periodic multiscale data

• Let $\mathbf{x}^f_{\mathbf{j}} = \mathbf{j}h$ be the fine grid, and $\mathbf{x}^c_{\mathbf{j}} = \mathbf{j}H$ be the coarse grid. We write

$$\mathbf{u}_0(\mathbf{x}_{\mathbf{j}}) = \mathbf{u}_0(\mathbf{x}^c_{\mathbf{j}}, \mathbf{x}^f_{\mathbf{j}}).$$

• Rescale the sub-grid cell problem of size H to a unit domain of order one. Let $\mathbf{y} = \mathbf{x}/H$,

$$\mathbf{u}_0(\mathbf{x}_{\mathbf{j}}) = \mathbf{u}_0(\mathbf{x}^c_{\mathbf{j}}, \mathbf{y}_{\mathbf{j}}/\epsilon),$$

where $\mathbf{y}_{\mathbf{j}} = \mathbf{j}Hh$, and $\epsilon = H$

• Further, we define an average operator:

$$\langle \mathbf{u}_0 \rangle = \int_{[-\frac{1}{2},\frac{1}{2}]^3} \mathbf{u}_0(\mathbf{x}, \mathbf{y}) d\mathbf{y}.$$

And decompose initial velocity field $\mathbf{u}_0$ as $\mathbf{u}_0 = U(\mathbf{x}) + \mathbf{W}(\mathbf{x}, \mathbf{y})$ with $\langle \mathbf{W} \rangle = \mathbf{0}$.

• We can use basically the same multiscale analysis to derive the homogenized equation.

• We make the same expansions for $\mathbf{u}^\epsilon$ and p^ϵ

$$\mathbf{u}^\epsilon(t,\mathbf{x}) = \mathbf{u}(t,\mathbf{x}) + \mathbf{w}(t,\mathbf{x},\tau,\mathbf{y}) + O(\epsilon),$$
$$p^\epsilon(t,\mathbf{x}) = p(t,\mathbf{x}) + q(t,\mathbf{x},\tau,\mathbf{y}) + O(\epsilon),$$

where $\mathbf{w}$ and q have zero mean, $\tau = t/\epsilon$ and $\mathbf{y} = \theta^\epsilon(t,\mathbf{x})/\epsilon$.

• The map θ^ϵ is given by

$$\theta^\epsilon = \theta(t,\mathbf{x}) + \epsilon\theta^{(1)}(t,\mathbf{x},\frac{t}{\epsilon},\frac{\theta^\epsilon}{\epsilon}) + O(\epsilon^2),$$

and satisfies

$$\theta^\epsilon_t + \mathbf{u}^\epsilon \cdot \nabla\theta^\epsilon = \mathbf{0}, \quad \theta^\epsilon(0,\mathbf{x}) = \mathbf{x}.$$

Remark on more general multiscale solutions

- The homogenization theory presented here can be used as a guideline to design a multiscale method for more general multiscale solutions.

- Small scales for velocity $\mathbf{w}$ and pressure q are strongly localized.

- For general data, a local cell problem can be constructed to supply small scale information for the coarse grid model (as in MsFEM).

- Reynolds stress term $\langle \mathbf{ww} \rangle$ is expected to reach local statistical equilibrium very fast in τ.

Numerical Experiments

We use direct numerical simulation (DNS) to check the accuracy of our multiscale analysis in 2-D.

The DNS is performed using 512 by 512 mesh. The multiscale computations use 64 by 64 on the coarse grid, and 32 by 32 for the subgrid.

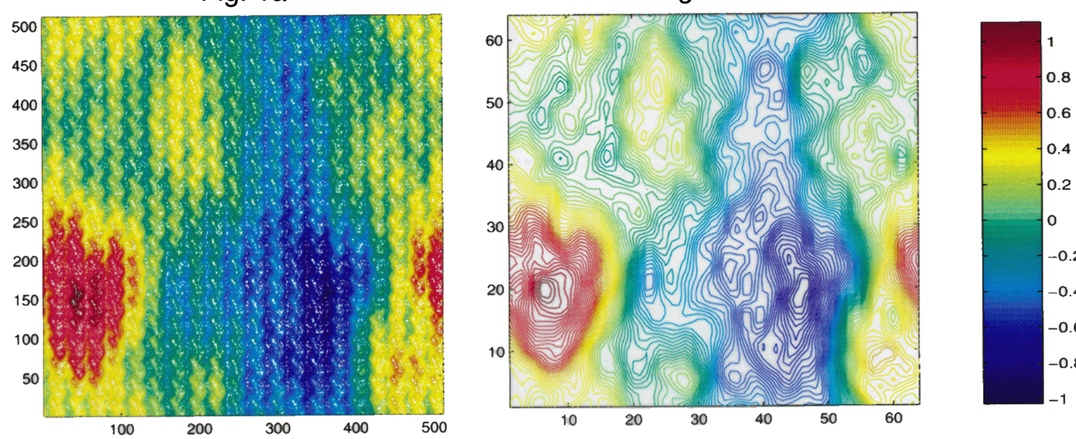
Figure 1 shows the initial horizontal velocity in fine and coarse grid. It has no scale separation.

Figure 2 shows the horizontal velocity at $t = 0.5$ in fine and coarse grid.

Figure 3 shows the averaged velocity at $t = 1$ obtained by DNS and by homogenization respectively. The agreements are very good.

Fig. 1a

Fig. 1b

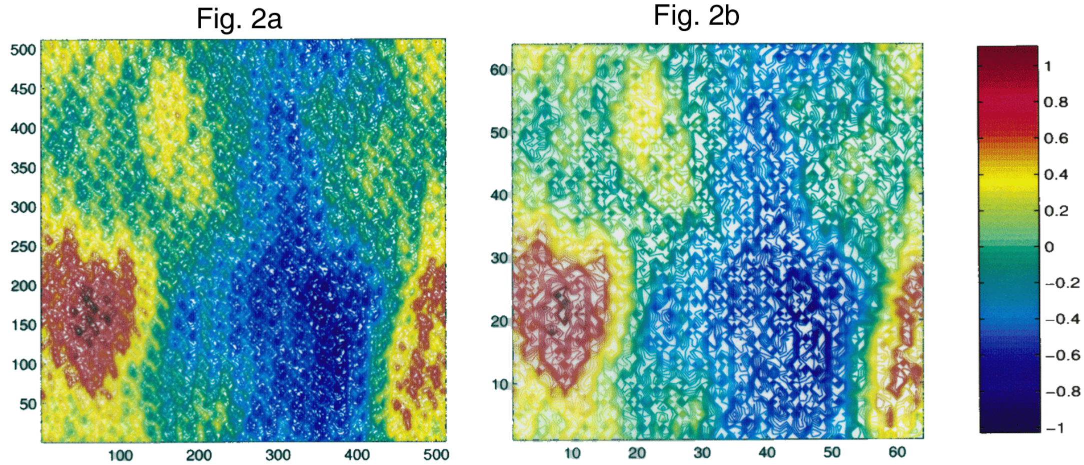


t=0 U+W (fine grid)

t=0 U (coarse grid)

Fig. 2a

Fig. 2b

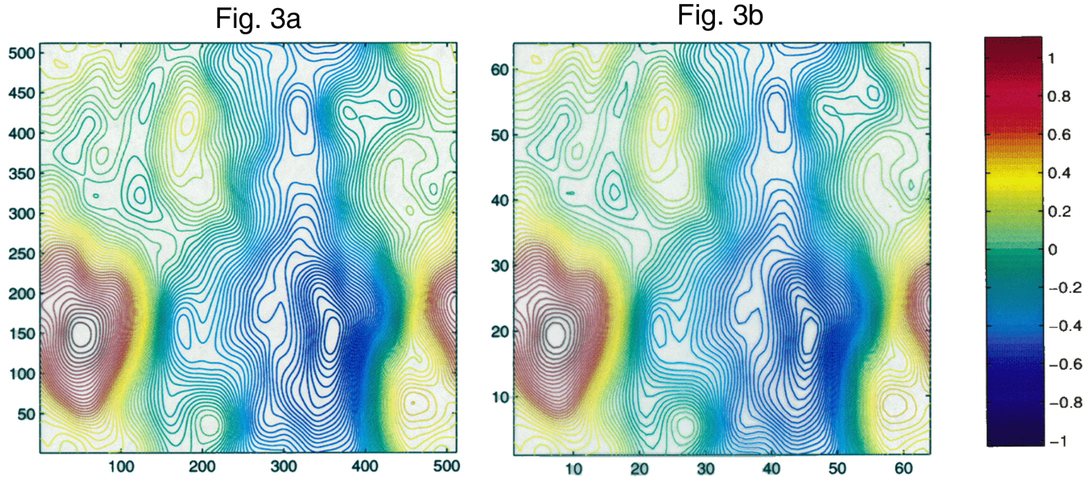


t=0.5 U+W (DNS, fine grid)

t=0.5 U+W (interpolated on coarse grid)

Fig. 3a

Fig. 3b



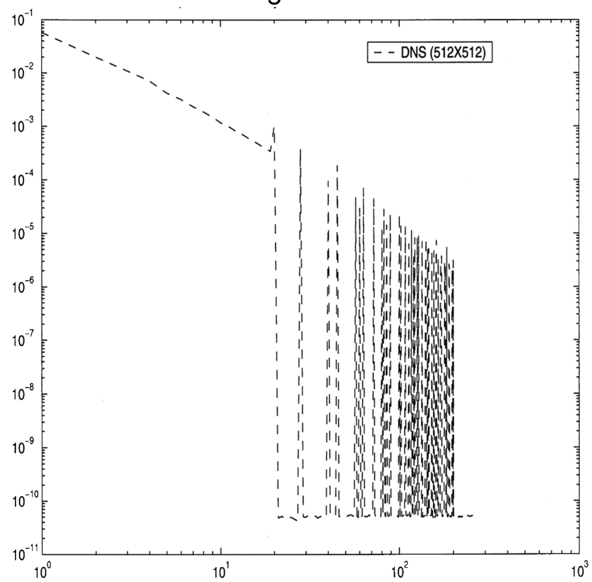
t=1.0 mean flow U (DNS, fine grid)
filter k=0.01

t=1.0 mean flow U (coarse grid)
filter k=0.01

Figure 4 shows the Fourier spectrum at $t = 0$ and at $t = 0.5$ respectively. The spectrum from homogenization agrees very well with that from DNS.

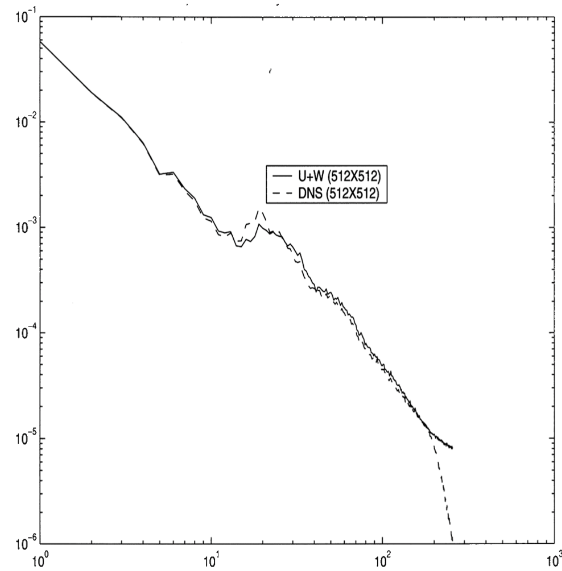
Figure 5 shows the cross sections of the averaged horizontal velocity at $t = 1$ obtained by DNS and by homogenization respectively.

Fig. 4a



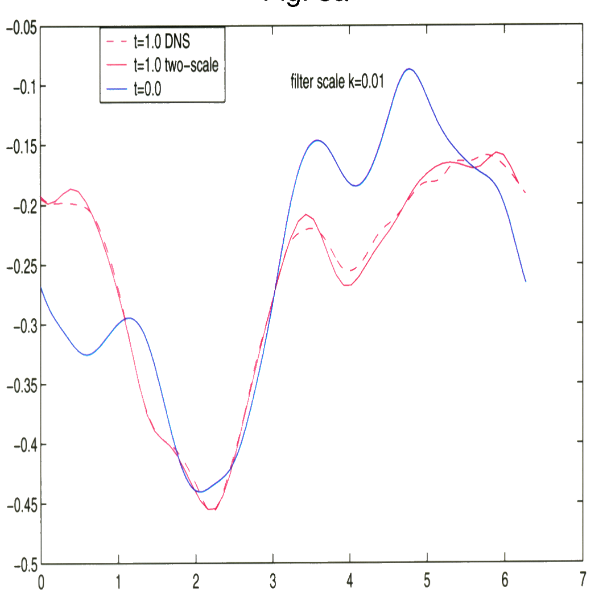
t=0 spectrum of velocity

Fig. 4b



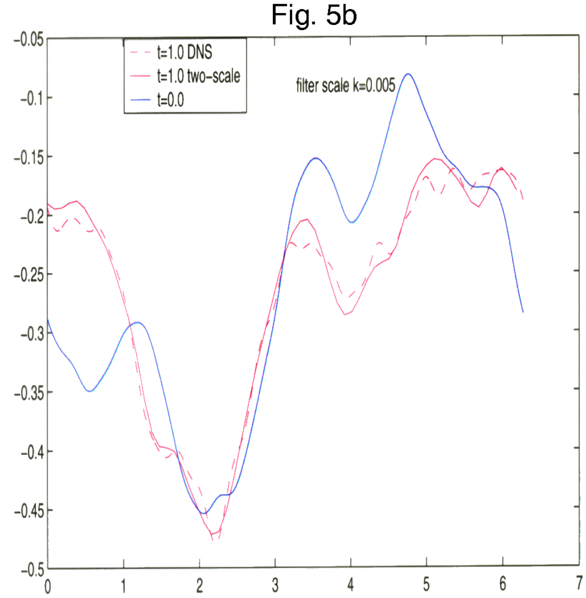
t=0.5 spectrum of velocity

Fig. 5a



t=1.0, cross-section of mean flow U
filter scale = 0.01

Fig. 5b



t=1.0, cross-section of mean flow U
filter scale = 0.005

Conclusions

• The multiscale analysis provides a useful guideline to design effective coarse grid methods for incompressible flow.

• When the flow is fully mixed, we expect that the Reynolds stress term, i.e. $\langle \mathbf{ww} \rangle$, will reach to a statistical equilibrium relatively fast.

• As a consequence, we may need to solve for the cell problem in τ for a small number of time steps.

• For homogeneous flow, it should be sufficient to solve one or a few representative cell problems to compute the Reynolds stress.

• This would offer an efficient coarse method that couples the large and small scales dynamically at a cost comparable to LES.